

Updating Directed Minimum Cost Spanning Trees

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joint work with

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Outline

- 1 Introduction
- 2 Complexity of Updates
- 3 A dynamic algorithm
 - The ATree structure
 - Edge Deletion
 - Edge Insertion
 - Implementation and Complexity
- 4 Experiments

Introduction

A DMST is a min-cost subset of (directed) edges $T \subseteq E$ such that:

- 1 T is acyclic
- 2 $(u, v), (x, y) \in T \Rightarrow v \neq y$
- 3 T is maximal with respect to 1 and 2.

Branchings vs. Trees

- G strongly connected \Rightarrow a DMST T is a tree of $n - 1$ edges.
- Otherwise T may be forest of directed trees called branching.
- We can always augment G with infinite cost edges to make it strongly connected.

Algorithms

For a digraph $G(V, E)$, $|V| = n$, $|E| = m$:

- 1 Same algorithm (*Edmonds' algorithm*) independently by:
Edmonds 1967, Chu & Liu 1965, Bock 1971: $O(mn)$
- 2 Best known implementation for general digraphs:
Gabow, Galil, Spencer, Tarjan 1986: $O(m + n \log n)$
- 3 Best known implementation for sparse digraphs ($O(n)$ edges):
Mendelson, Tarjan, Thorup, Zwick, 2004: $O(m \log \log n)$

Dynamic Updates

Efficiently updating the DMST $T \subseteq E$ of a digraph $G(V, E)$ when:

- 1 An edge is deleted from the graph.
- 2 An edge is inserted to the graph.

... in less time than re-evaluating it from scratch.

The dynamic complexity of DMST is not known.

For MST see [Holm, de Lichtenberg, Thorup, JACM 2001](#)

Is it Possible?

DMST on a Directed Acyclic Graph:

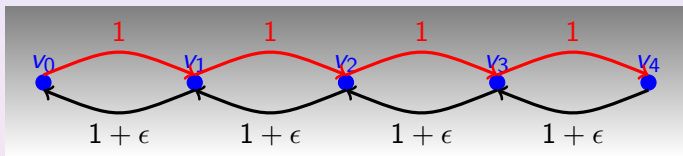
For each $v \in V$ choose the min-cost incoming edge.

Cycles will not occur.

Verification Time on a DAG

- $cost(T)$ minimum iff $\forall (u, v) \in T$ min-cost edge entering v .
- This amounts testing $O(m)$ inequalities.
- $\Omega(m)$ time is required (Rabin 1972).

A weak lower bound

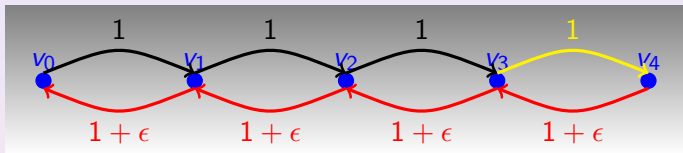


Sensitivity to edge operations

Removal of (v_3, v_4) from the digraph and the **DMST**...

Theorem

A weak lower bound



Sensitivity to edge operations

... causes the **DMST** to change entirely! At least verification time is required to update: $\Omega(n^2)$ for dense digraphs.

Theorem

A dynamic algorithm retaining and using only the DMST itself requires $\Omega(m)$ time, which is $\Omega(n^2)$ for dense digraphs.

Note: $O(\log n)$ Update Time on DAGs

- For each vertex v store its incoming edges in a FHeap $H(v)$.
- Deletion of $(u, v) \in E$ from $G(V, E)$:
 - 1 $(u, v) \in T$: *extract* least-cost e from $H(v)$ to replace (u, v) .
 - 2 $(u, v) \notin T$: *decrease-cost* to $-\infty$ of (u, v) in $H(v)$ and *extract* (u, v) from $H(v)$.
- Insertion of (u, v) to E :
 - 1 $\exists(x, v) \in T$ with $cost(x, v) \leq cost(u, v)$: *insert* (u, v) in $H(v)$.
 - 2 $\exists(x, v) \in T$ with $cost(x, v) > cost(u, v)$: replace (x, v) with (u, v) and *insert* (x, v) in $H(v)$.

An algorithm for dynamic updates

A *dynamization* of Edmonds' algorithm:

- Encode output into an Augmented Tree (ATree) structure.
- Modify the data structure per edge operation.
- Identify maximal portion of *unaffected* output to maintain.
- Execute a modified implementation to augment the maintained output.

Edmonds' Algorithm

- 1 Set $v = v_0$.
- 2 Repeat until a single vertex remains:
 - 1 Select the min-cost edge entering v , say (u, v) .
 - 2 For each edge (x, v) set $cost(x, v) \leftarrow cost(x, v) - cost(u, v)$.
 - 3 Contract dicycle (if any) due to (u, v) , to vertex u .
 - 4 Set $v = u$.
- 3 Remove redundant selected edges.

Edmonds' Algorithm

- 1 Set $v = v_0$.
- 2 Repeat until a single vertex remains: [$O(m + n \log n)$ Time]
 - 1 Select the min-cost edge entering v , say (u, v) .
 - 2 For each edge (x, v) set $cost(x, v) \leftarrow cost(x, v) - cost(u, v)$.
 - 3 Contract dicycle (if any) due to (u, v) , to vertex u .
 - 4 Set $v = u$.
- 3 Remove redundant selected edges. [$O(n)$ Time]

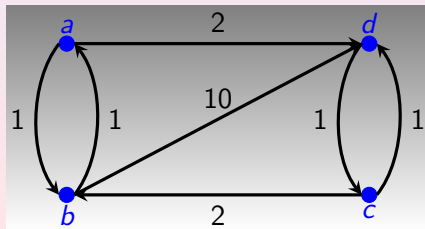
We *dynamize* the loop

Example on Edmonds' Algorithm

Next Step

Take the least cost edge entering vertex a .

That is edge (b, a) .



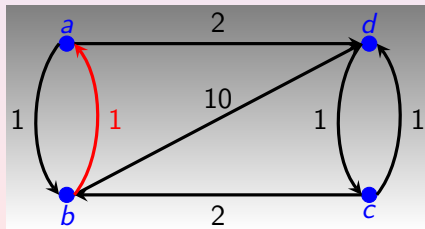
Example on Edmonds' Algorithm

Next Step

Take the least cost edge entering vertex b .

That is edge (a, b) .

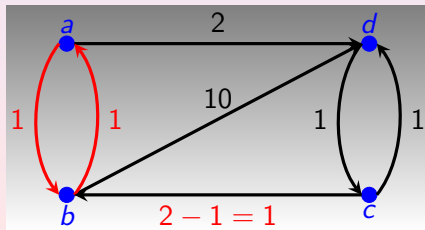
Reduce the cost of (c, b) to $2 - \text{cost}(a, b) = 2 - 1 = 1$.



Example on Edmonds' Algorithm

Next Step

Contract cycle $a \rightarrow b \rightarrow a$ into a vertex A .

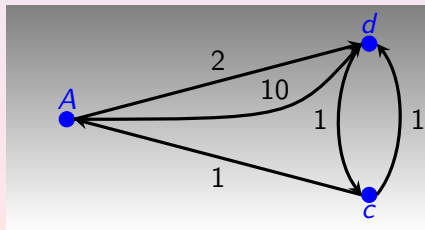


Example on Edmonds' Algorithm

Next Step

Take the least cost edge entering vertex A .

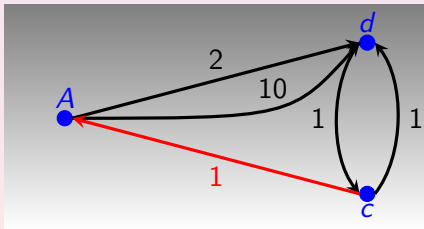
That is edge (c, A) (which was edge (c, b)).



Example on Edmonds' Algorithm

Next Step

Take the least cost edge entering c .
That is edge (d, c) .



Example on Edmonds' Algorithm

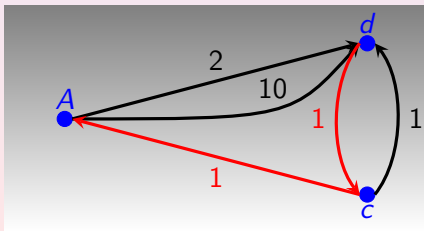
Next Step

Take the least cost edge entering d .

That is edge (c, d) .

Reduce the cost of (a, d) to $2 - \text{cost}(c, d) = 2 - 1 = 1$.

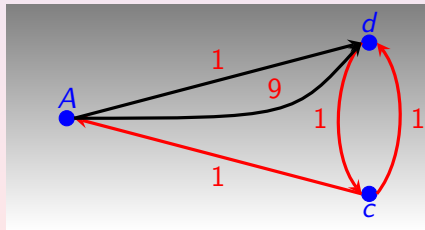
Reduce the cost of (b, d) to $10 - \text{cost}(c, d) = 10 - 1 = 9$.



Example on Edmonds' Algorithm

Next Step

Contract cycle $c \rightarrow d \rightarrow c$ into a vertex B .

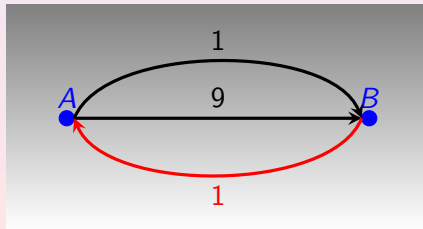


Example on Edmonds' Algorithm

Next Step

Take the least cost edge entering B .

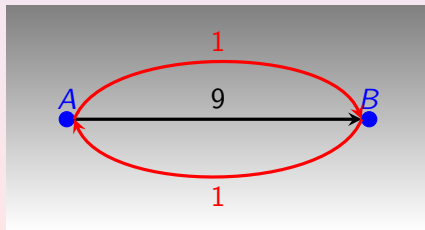
That is edge (A, B) (which was edge (a, d)).



Example on Edmonds' Algorithm

Next Step

Contract cycle $A \rightarrow B \rightarrow A$ into a vertex C .

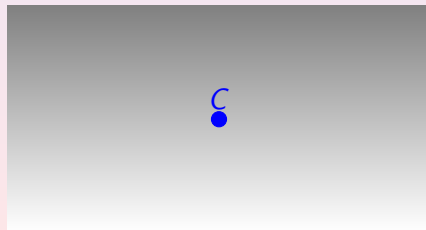


Example on Edmonds' Algorithm

Next Step

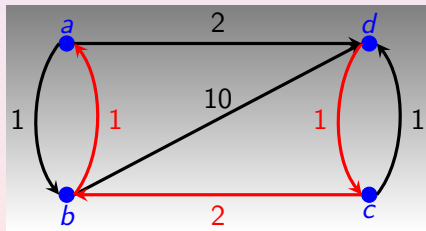
End of loop.

Removal of redundant edges produces the DMST.



Example on Edmonds' Algorithm

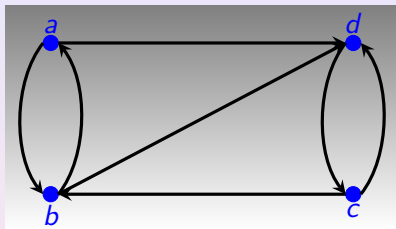
Next Step



Encode the output of the loop into a hierarchical structure (ATree):

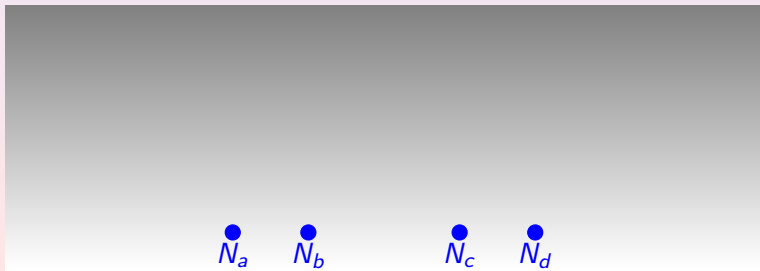
- 1 *Leaf Nodes*: one for each vertex of the digraph.
- 2 *Internal Nodes*: Represent vertices emerging from contractions.
- 3 Each node is labelled with the edge selected for the represented vertex.
- 4 Each internal node has an associated list of edges absorbed due to contraction.

Bottom-Up Construction of an ATree (Example)

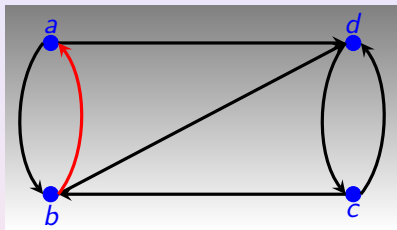


Next Step

Label N_a with (b, a) .

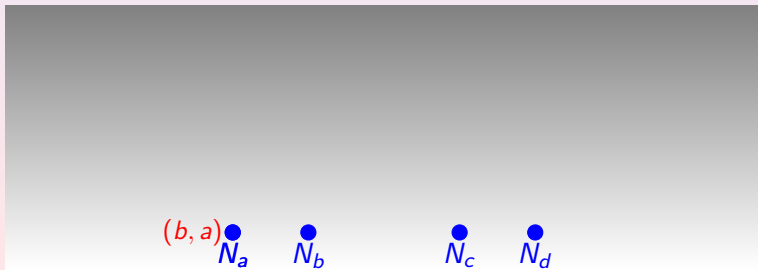


Bottom-Up Construction of an ATree (Example)

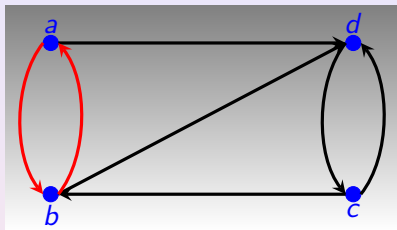


Next Step

Label N_b with (a, b) .

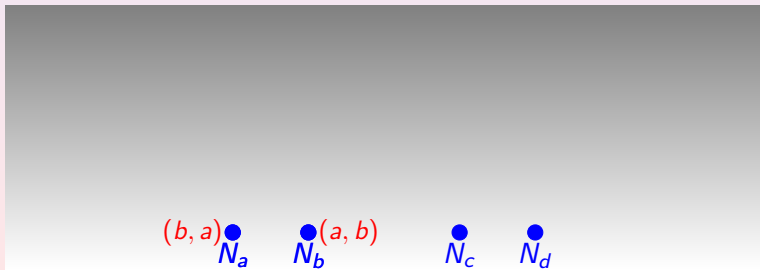


Bottom-Up Construction of an ATree (Example)

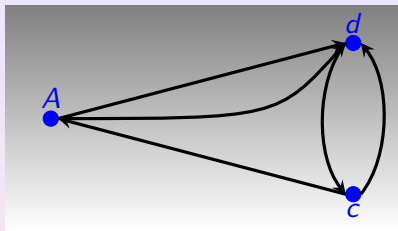


Next Step

Create N_A , parent of N_a, N_b .

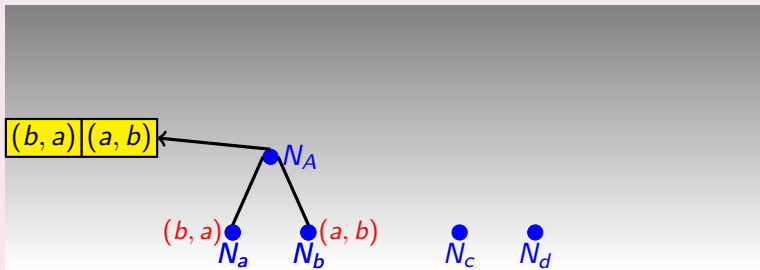


Bottom-Up Construction of an ATree (Example)

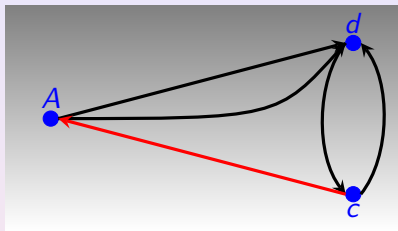


Next Step

Label N_A with (c, b) .

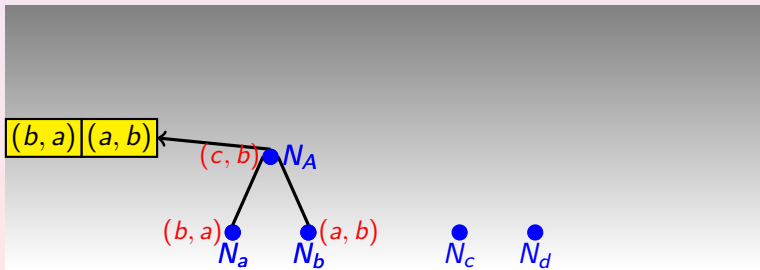


Bottom-Up Construction of an ATree (Example)

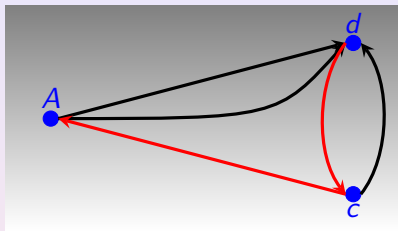


Next Step

Label c with (d, c) .

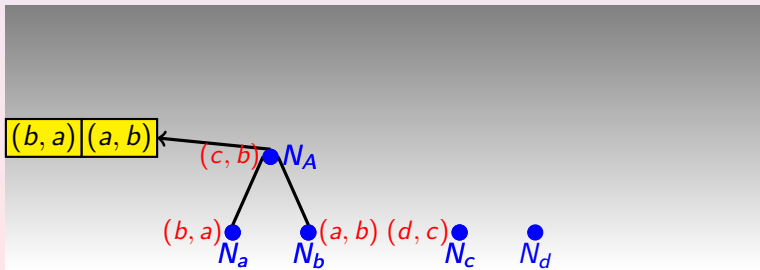


Bottom-Up Construction of an ATree (Example)

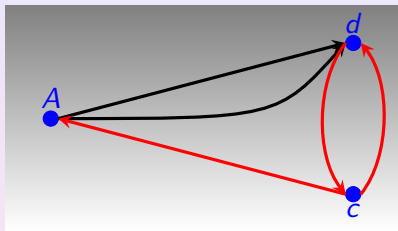


Next Step

Label d with (c, d) .

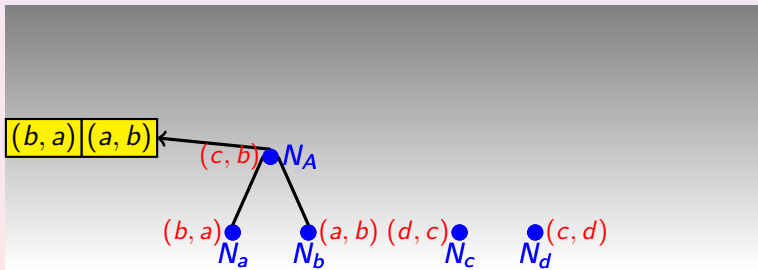


Bottom-Up Construction of an ATree (Example)

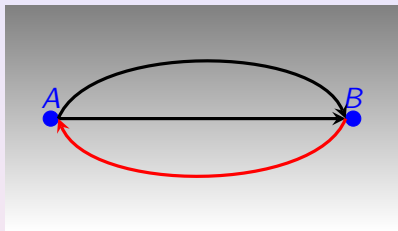


Next Step

Create N_B , parent of N_c, N_d .

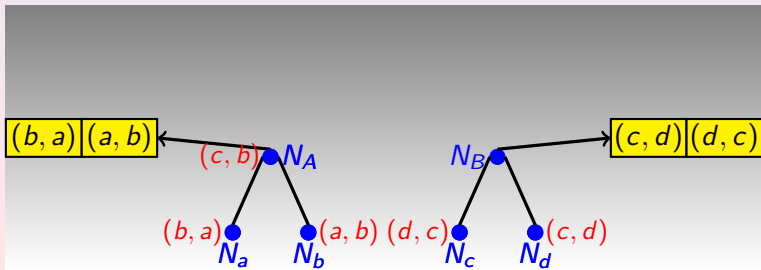


Bottom-Up Construction of an ATree (Example)

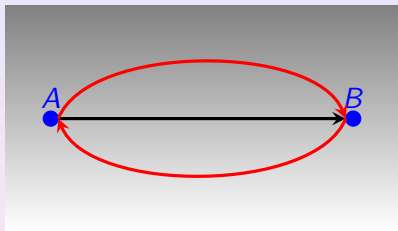


Next Step

Label N_B with (a, d) .

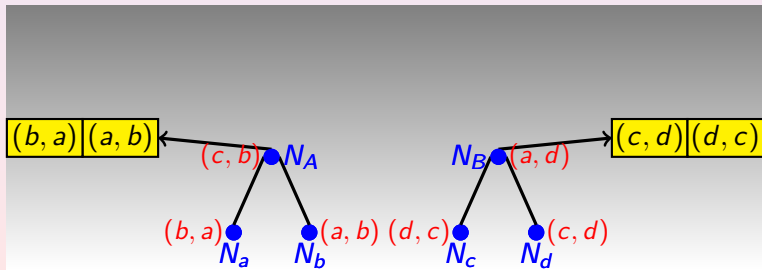


Bottom-Up Construction of an ATree (Example)

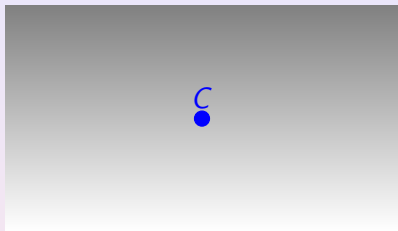


Next Step

Create N_C , parent of N_A , N_B .

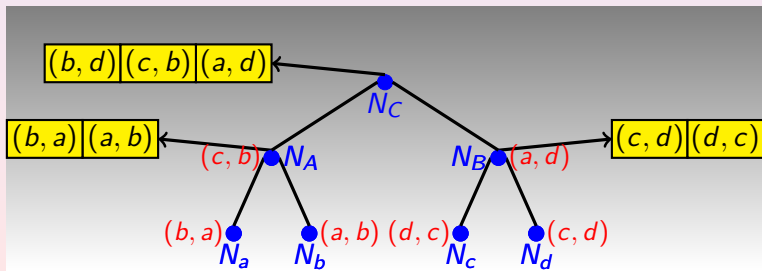


Bottom-Up Construction of an ATree (Example)



Next Step

The *labelset* \mathcal{L} of the ATree is the set of edges label its nodes.

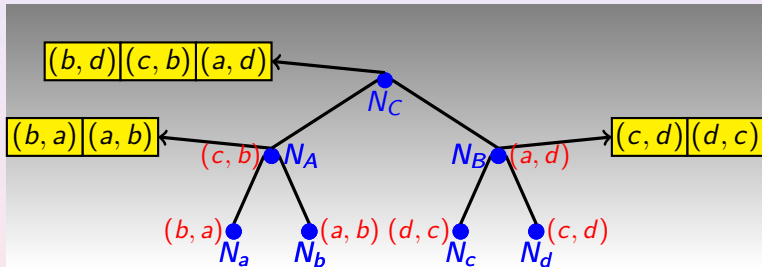


Deleting an edge

Let e be the deleted edge. Two cases must be considered:

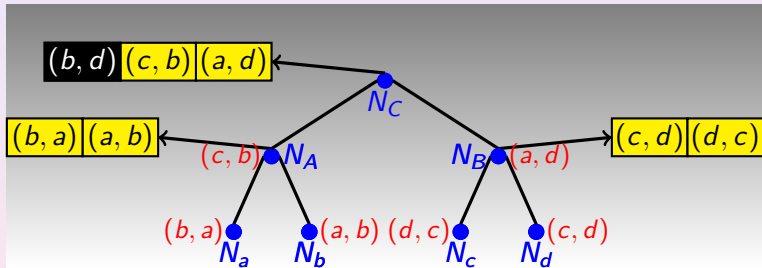
- $e \notin \mathcal{L}$: Update the ATree and Stop.
- $e \in \mathcal{L}$:
 - 1 Maintain a maximal portion of the ATree, unaffected by the deletion.
 - 2 This represents a maximally contracted digraph G' .
 - 3 Initialize and execute Edmonds' algorithm on G' .

Edge Deletion Example



Case 1: Deleting (b, d) not in the labelset \mathcal{L}

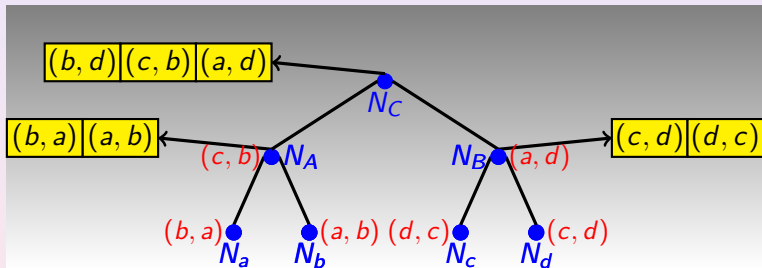
Edge Deletion Example



Case 1: Deleting (b, d) not in the labelset \mathcal{L}

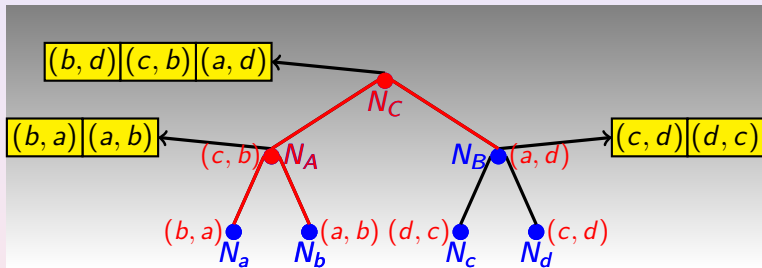
Remove (b, d) from the list containing it.

Edge Deletion Example



Case 2: Deleting (a, b) from the labelset \mathcal{L}

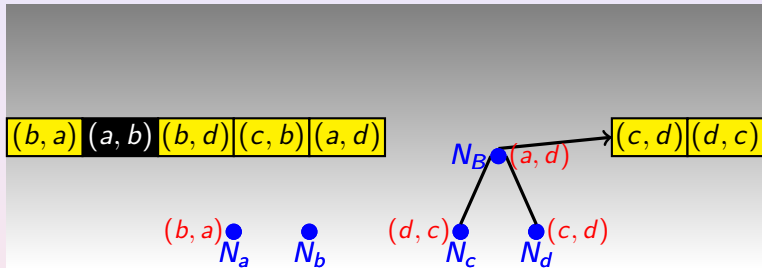
Edge Deletion Example



Case 2: Deleting (a, b) from the labelset \mathcal{L}

Remove the path from N_b to the root, N_C , and unite the orphaned associated lists.

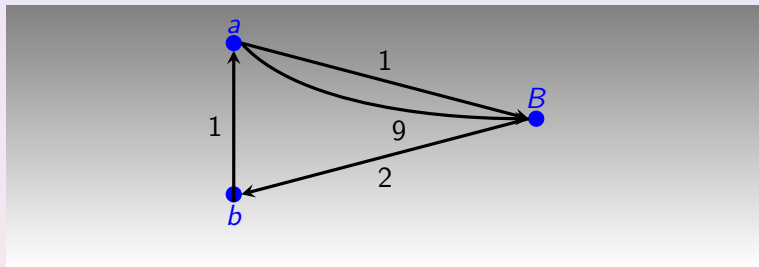
Edge Deletion Example



Case 2: Deleting (a, b) from the labelset \mathcal{L}

- Roots $\{N_a, N_b, N_B\}$ are vertex set of maintained contracted digraph.
- List union of removed nodes (but (a, b)) is the edge set.

Edge Deletion Example



A maximally contracted digraph unaffected by deletion

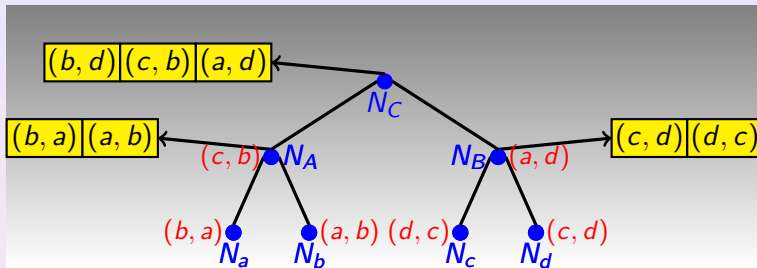
- Cost assignment on edges can be done in $O(n)$ time.
- Execute Edmonds' algorithm on this digraph.

Inserting an Edge

Two cases must be considered:

- $cost(T)$ is not affected: update the ATree and stop.
- The newly inserted edge reduces $cost(T)$.
 - 1 e substitutes some $e' \in \mathcal{L}$.
 - 2 Engage *virtual* deletion of e' .
 - 3 Initialize and execute Edmonds' algorithm on $G'(V', E' \cup \{e\})$.

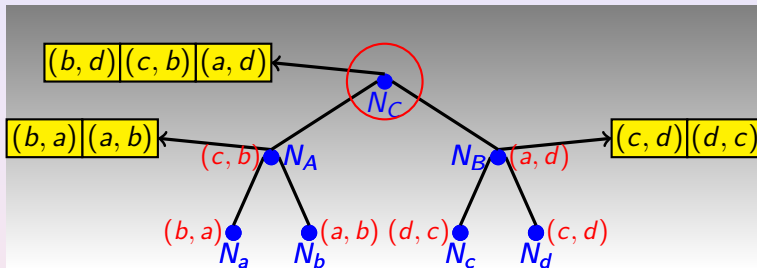
Edge Insertion Example



Insertion of edge (a, c) at cost 10.

- 1 Find *Least Common Ancestor* (LCA) of N_a and N_c .

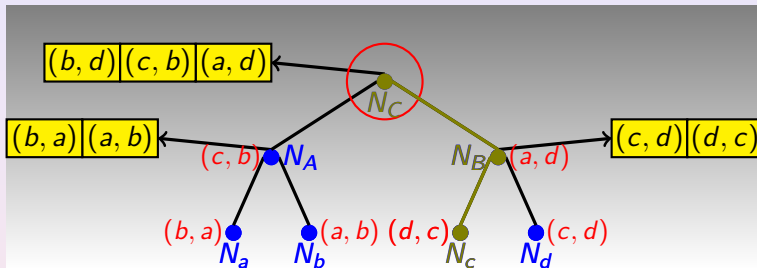
Edge Insertion Example



Insertion of edge (a, c) at cost 10.

- 1
- 2 Take the subtree rooted at the found LCA (N_C).

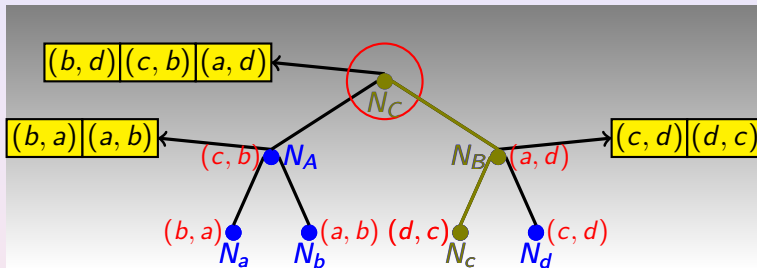
Edge Insertion Example



Insertion of edge (a, c) at cost 10.

- 1
- 2
- 3 Follow the path $N_C \cdots N_C$, comparing $cost(a, c)$ with the cost of the labelling edges of the nodes in the order visited.

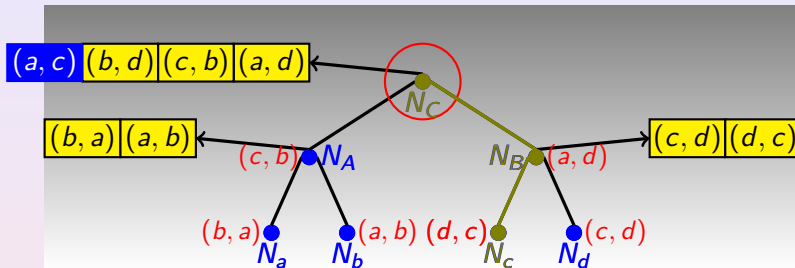
Edge Insertion Example



Case 1: (a, c) costs less than (d, c) or than (a, d) .

- Engage a *virtual* deletion of (d, c) or (a, d) respectively.

Edge Insertion Example



Case 2: (a, c) cannot replace an edge from labelset

- 1
- 2 Insert (a, c) into the list of the LCA

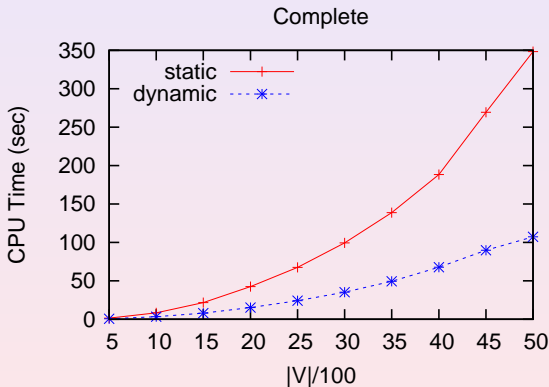
Technical Details

- All operations on ATree structure take $O(n)$ time.
- Implementation relies on ideas of Gabow et al., 1986.
- For sparse digraphs use heaps of Mendelson et al. 2004.
- Output Complexity (in terms of minimal output subset needing update):
 - ρ denotes vertex set of maximally contracted digraph.
 - Can be shown to be $O(n + ||\rho|| + |\rho| \log |\rho|)$.
 - $||\rho||$ denotes edge set entering vertices of ρ .
 - Worst case equal to complexity of re-evaluation.

Experimental Evaluation

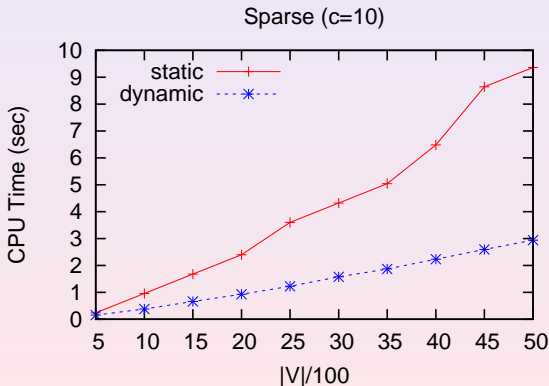
- Random Digraphs of orders 500 – 5000, step 500.
 - 1 Dense Digraphs: $d \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$.
 - 2 Sparse Digraphs: $d = \frac{c}{n-1}$, $c \in \{10, 20, 30, 40, 50\}$.
- Sparse digraphs with embedded clique of increasing order.
- Evaluating updates in comparison to static DMST re-evaluation.
- Average time over 10^4 deletions/insertions decided with 0.5.

Performance on Complete Digraphs



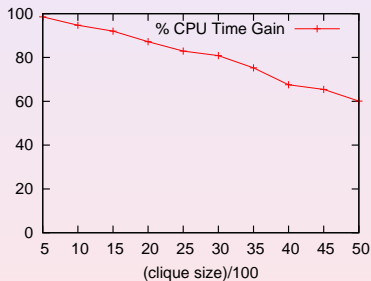
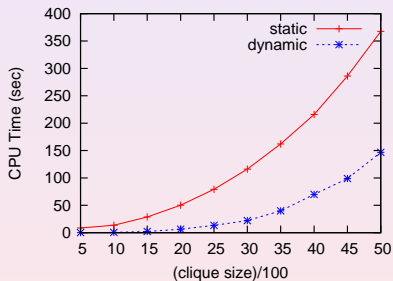
Over 60% relative gain in execution time in dense digraphs.

Performance on Sparse Digraphs



Increasing gain with digraph order in sparse digraphs: 60% to 80%

Embedded Cliques



As the clique grows the performance falls from a 95% to a 60%.

Conclusions

- A (weak) lower bound on the complexity of DMST updates.
- A dynamic algorithm studied in terms of output complexity and experimentally.
- Achieving a factor > 2 speedup on dense digraphs.
- Achieving huge speedup on sparse digraphs:
 - For DAGs our scheme reduces to the $O(\log n)$ update time
 - What about average case complexity for sparse digraphs?
- The complexity of updates is still open.

Platform of Experimentation

- Standard C++ implementation, gcc 3.2
- P4 2.6 GHz, 512 MB Main memory
- Linux Kernel 2.6