VIRTUES OF THE HAVERSINE

There is nothing fundamental or 'sacred' about the trigonometric functions we call the sine, cosine, and tangent. Their reciprocals (the cosecant, secant, and cotangent, respectively) were once widely taught and tabulated, probably because their use causes changes division into multiplication in certain formulas, and multiplying is easier with a pencil and paper.

Other geometrical relationships involving arcs and triangles could have been adopted and indeed have been in the past. The Greek astronomer Claudius Ptolemy, for example, used the chord, which is twice the sine of half the angle. Another archaic trig function is the versed sine, which equals 1 minus the cosine. Furthermore, half the versed sine, contracted to hav-sine, was once widely used by navigators on the high seas.

I would like to make a plea for reinstating the hav-sine, perhaps as a labeled calculator key or as part of the function library in an extended Fortran or Basic. Unlike some of the old trig functions, whose use merely facilitated calculations done by longhand or with logarithms, the hav-sine is a great aid in certain astronomical calculations involving small angles. It can preserve significant digits.

A perfect example of this arises in calculating the angular separation $s$ of one star or planet from another. Suppose the right ascension and declination of the first object are $\alpha_1, \delta_1$, and those of the second are $\alpha_2, \delta_2$. The standard formula given in books on spherical astronomy is this:

$$
\cos s = \sin \delta_1 \sin \delta_2 + \cos \delta_1 \cos \delta_2 \cos (\alpha_2 - \alpha_1),
$$

where $\Delta \alpha$ is $\alpha_2 - \alpha_1$. It is fairly easy to write a Basic program to solve for $s$.

But even though the formula is truly exact, mathematically speaking, it becomes unstable in practice when the two objects are close together, a situation that arises with appulses (when a planet passes close to a star) or in work on double stars. The difficulty is illustrated by this table:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\Delta \alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>$1^\circ$</td>
</tr>
<tr>
<td>$0.98614698$</td>
<td>$0.9999949577$</td>
</tr>
<tr>
<td>$1^\circ$</td>
<td>$0.9999999882$</td>
</tr>
<tr>
<td>$0.0000000000000000105$</td>
<td>$0.99999999999971$</td>
</tr>
</tbody>
</table>

The nines after the decimal point are carried as significant digits by a computer, and accuracy is very poor when a small angle must be evaluated from its cosine. Indeed, with Basics that work to seven-digit accuracy, and stars closer than a minute of arc, it may happen that the computer will announce the separation to be zero!

To avoid this preposterous result, we could switch to a machine that maintains more digits internally. But this really doesn't eliminate the problem; the machine will still fail at some smaller angle. It's better, when dealing with very small separations, to abandon the cosine formula in favor of one based on the Pythagorean theorem. Think of one star as being a small amount "up and over" from the other one. Their difference in declination, expressed in seconds of arc, forms one side of a small right triangle. The difference in right ascension, expressed in seconds of time and multiplied by $15^\circ$ to convert it to arc seconds too, forms the base of the triangle. Then the angular separation is simply the hypotenuse of the triangle, or the square root of the sum of the squares of the other two sides.

This Pythagorean approach is not rigorously exact, because we are dealing with the curved sky rather than a plane right triangle. But the smaller the separation, the more accurate it becomes. It starts to fail with angles much larger than $1^\circ$ or near the celestial poles. There is always some doubt, in practice, as to the particular separation at which it becomes less accurate than the cosine formula.

Altogether the best way to calculate angular separation is to transform the cosine formula into one that uses hav-sines. For any angle $\theta$, $\text{hav}\theta = (1 - \cos \theta)/2$, and it is not hard to show that the cosine formula for angular separation is precisely equivalent to this one:

$$
\text{hav}\ s = \text{hav}\ \Delta \alpha + \cos \delta_1 \cos \delta_2 \text{hav}\ \Delta \alpha.
$$

But how do we use this, when computer languages don't have the hav-sine function? We can still get the benefit of the idea with the help of another identity, $\text{hav}\ \theta = \sin^2(\theta/2)$. Listing 1 uses this relationship to convert back and forth between sines and hav-sines, and the result is a program that computes angular separations accurately and without any difficulty for angles from nearly $180^\circ$ all the way down to exactly $0^\circ$.

Let's test the program on some demanding cases. In the following example, we'll try it out on the famous wide double star in Ursa Major, Mizar and Alcor. According to the Boss General Catalogue, Mizar's 1950.0 astrometric position is $13^\circ 21' 54.54353$, $+55^\circ 11' 09.24''$, while Alcor lies at $13^\circ 23' 13.544$, $+55^\circ 14' 52.78''$.

$$
\text{RUN "ANGSEP"}
\text{FIRST STAR --}
\text{R A (h,m,s) ? 13, 21, 54.54353}
\text{DEC (d,m,s) ? 55, 11, 09.24}
\text{SECOND STAR --}
\text{R A (h,m,s) ? 13, 23, 13.544}
\text{DEC (d,m,s) ? 55, 14, 52.78}
\text{SEPARATION --}
\text{IN DEGREES : .196849}
\text{IN ARC SEC : 708.655}
\text{READY}
$$

This sample was run on a Radio Shack TRS-80 Model III, and the result is quite close to the correct value, $708.69''$. Unless the answer you get from this or any other program is similar, something's wrong. (On the TRS-80 the cosine formula gives $722.76''$, an erroneous result produced by the indeterminacy for small angles mentioned earlier.)

For another useful test, try a pair of hypothetical stars. Both have a declination of $+20^\circ$, but their right ascensions are $0^\circ$ and $12^\circ$, respectively. A little head-scratching shows that the shortest arc between them passes through the pole, and their separation must be $140^\circ$ exactly. The computer should agree.

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