Linear Program in standard form

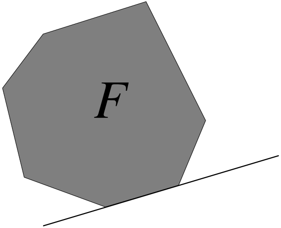
Maximize

$$\sum_{j=1}^{n} c_j x_j$$

Subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \quad i = 1, 2, \ldots, m \\ x_j &\geq 0, \quad j = 1, 2, \ldots, n \end{aligned}$$

Some intuition



F is the intersection of **Halfspaces**, a **Convex Polyhedron**.

Intuitive fact 1: A linear function $\langle c, x \rangle$ which attains its minimum in F does so in a **corner** of F.

Intuitive fact 2: If v is a corner which does *not* minimize $\langle c, x \rangle$, an improvement can be made by walking along an **edge** of F next to v.

Local Search Pattern

```
LocalSearch(ProblemInstance p)
    x := feasible solution to p
    while ∃y ∈ N(x) : y better than x do
        x := y
    od
    return x
end
```

$N(x)$ is the *neighborhood* of x.

Simplex algorithm (geometric sketch)

Maximize(LPinstance (A, b, c))

  $v :=$ some corner of F

  while there is an neighbor corner v' of v with better cost **do**

    $v := v'$

  od

  return v

end

The input

Maximize $5x_1 + 4x_2 + 3x_3$

Subject to

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &\leq 5 \\ 4x_1 + x_2 + 2x_3 &\leq 11 \\ 3x_1 + 4x_2 + 2x_3 &\leq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The input

Maximize $5x_1 + 4x_2 + 3x_3$

Subject to

$$\begin{aligned} x_4 + 2x_1 + 3x_2 + x_3 &= 5 \\ 4x_1 + x_2 + 2x_3 &\leq 11 \\ 3x_1 + 4x_2 + 2x_3 &\leq 8 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The input

Maximize $5x_1 + 4x_2 + 3x_3$

Subject to

$$\begin{aligned} x_4 + 2x_1 + 3x_2 + x_3 &= 5 \\ x_5 + 4x_1 + x_2 + 2x_3 &= 11 \\ 3x_1 + 4x_2 + 2x_3 &\leq 8 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

The input

Maximize $5x_1 + 4x_2 + 3x_3$

Subject to

$$
\begin{aligned}
x_4 + 2x_1 + 3x_2 + x_3 &= 5 \\
x_5 + 4x_1 + x_2 + 2x_3 &= 11 \\
x_6 + 3x_1 + 4x_2 + 2x_3 &= 8 \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

Equivalent system

Maximize z subject to $x_1, x_2, \ldots, x_6 \geq 0$ and

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 \end{array}$$

x_4, x_5, x_6 are called **slack variables**.

Equivalent system

Maximize z subject to $x_1, x_2, \ldots, x_6 \geq 0$ and

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 \end{array}$$

x_4, x_5, x_6 are called **slack variables**.

A feasible solution:
$x_1 = x_2 = x_3 = 0, x_4 = 5, x_5 = 11, x_6 = 8, z = 0.$

Want to maintain **invariant**: The current solution sets all variables on the right hand side to 0.

An improvement

An improvement can be made by increasing x_1 to 2.5:

$x_2 = x_3 = x_4 = 0, x_1 = 2.5, x_5 = 1, x_6 = 0.5, z = 12.5.$

$$\begin{array}{rcrcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
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x_4 should move right and x_1 should move left; we say we *pivot* the two variables. Let's rewrite.

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$$
\begin{array}{rcrcrcrcr}
x_1 & = & 2.5 & - & 1.5x_2 & - & 0.5x_3 & - & 0.5x_4 \\
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x_4 should move right and x_1 should move left; we say we *pivot* the two variables. Let's rewrite. Substitute the expression for x_1 in the other equations.

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x_4 should move right and x_1 should move left; we say we *pivot* the two variables. Let's rewrite. Substitute the expression for x_1 in the other equations.

Stuck?

Another improvement

Another improvement can be made by raising x_3 to 1.

$x_2 = x_4 = x_6 = 0, x_3 = 1, x_1 = 2, x_5 = 1, z = 13.$

$$
\begin{array}{rcrcrcrcr}
x_1 & = & 2.5 & - & 1.5x_2 & - & 0.5x_3 & - & 0.5x_4 \\
x_5 & = & 1 & + & 5x_2 & & & + & 2x_4 \\
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x_6 should move right and x_3 should move left. Let's rewrite.

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$x_2 = x_4 = x_6 = 0, x_3 = 1, x_1 = 2, x_5 = 1, z = 13.$

$$\begin{array}{rcrcrcrcr} x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\ x_5 & = & 1 & + & 5x_2 & & & + & 2x_4 \\ \color{red}{x_3} & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\ z & = & 12.5 & - & 3.5x_2 & + & 0.5\color{red}{x_3} & - & 2.5x_4 \end{array}$$

x_6 should move right and x_3 should move left. Let's rewrite. Substitute the expression for x_3 in the other equations.

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$$
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x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\
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x_6 should move right and x_3 should move left. Let's rewrite. Substitute the expression for x_3 in the other equations.

Stuck?

The optimal solution has been found

The solution found is optimal: No solution with $x_1, x_2, \ldots, x_6 \geq 0$ can yield a value of $z = 13 - 3x_2 - x_4 - x_6$ better than 13.

Dictionaries

Given an LP instance in standard form with m constraints and n variables and with feasible origin.

A **dictionary** for the instance is a system of equations on $m+n+1$ variables $x_1, x_2, \ldots, x_{m+n}, z$. The equations expresses $m+1$ of the variables, including z as linear functions of the rest. Maximizing z under the constraints given by the equations and $\forall i : x_i \geq 0$ must be **equivalent** to the original LP instance.

The left hand side variables (z excluded) are called the **basic** variables. The right hand side variables are called the **non-basic** variables. The set of basic variables is called the **basis**.

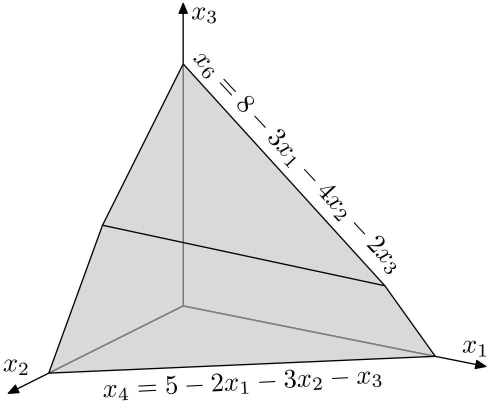
Dictionaries

The **basic solution** corresponding to the dictionary is obtained by assigning 0 to the nonbasic variables and letting the linear equations determine the values of the basic variables.

A basic solution corresponding to a dictionary is **feasible** if all values assigned to variables are at least 0 (i.e., if it is feasible for the original LP instance). A dictionary is **feasible** if its corresponding basic solution is.

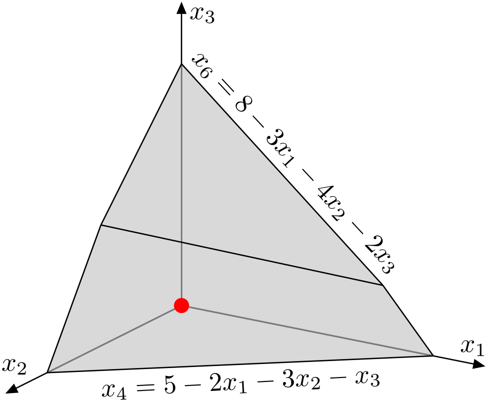
Invariant: Our current solution is a basic feasible solution represented by a dictionary. **Geometric Intuition**: Basic feasible solutions correspond to corner points of the polytope of feasible solutions.

Geometric intuition



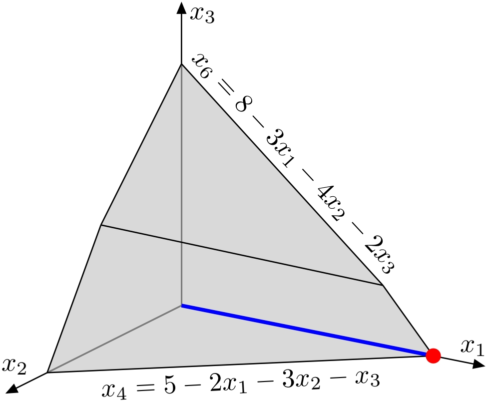
- A slack is a distance (not normalized)

Geometric intuition



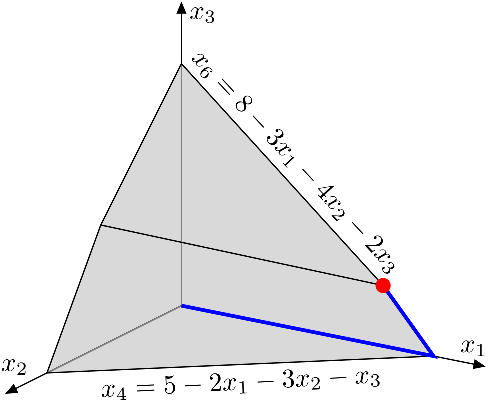
- A slack is a distance (not normalized)
- We started at $x_1 = x_2 = x_3 = 0$

Geometric intuition



- A slack is a distance (not normalized)
- We started at $x_1 = x_2 = x_3 = 0$
- Then moved to $x_2 = x_3 = x_4 = 0$

Geometric intuition



- A slack is a distance (not normalized)
- We started at $x_1 = x_2 = x_3 = 0$
- Then moved to $x_2 = x_3 = x_4 = 0$
- Terminated at $x_2 = x_4 = x_6 = 0$

Local Search checklist

Design:

1. How to find first feasible solution.
2. Define neigborhood structure N.
3. Strategy for choosing neighbor.

Analysis:

1. Partial correctness (Termination $\Rightarrow$ Correctness).
2. Termination.
3. Complexity.

Design 1: First feasible solution?

Maximize

$$\sum_{j=1}^{n} c_j x_j$$

Subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \quad i = 1, 2, \ldots, m \\ x_j &\geq 0, \quad j = 1, 2, \ldots, n \end{aligned}$$

The program has a **feasible origin** if $x = (0, 0, ..., 0)$ is a feasible solution. We *assume* for now that the program has feasible origin.

Design 1: First feasible dictionary

Our restriction: LP instances in standard form with feasible origin.

Introduce slack variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$. For $i = 1..m$, replace i'th constraint

$\sum_{j=1}^{n} a_{ij} x_j \leq b_i$

with

$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$

This yield our first feasible dictionary.

Design 2: Neighborhood structure

$D' \in N(D)$ iff:

- D' can be obtained from D by a pivot.
- D' is feasible.

Design 3: Which neighbor to pick?

Find variable x_i entering basis as any variable with strictly positive coefficient in expression for z.

Find a basic variable x_j whose equation most constrains increasing x_i.

Do the pivot on x_i and x_j: Rewrite expression for x_j as an expression for x_i. Substitute this expression for x_i in all other equations in D.

Special case: Unbounded dictionary

Maximize z subject to $x_1, x_2, \ldots, x_6 \geq 0$ and

$$\begin{array}{rcrcrcrcr}
x_4 & = & 5 & + & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & + & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & & & - & 4x_2 & - & 2x_3 \\
z & = & & & 5x_1 & + & 4x_2 & + & 3x_3
\end{array}$$

If x_i has positive or zero coefficient in all equations, report **Unbounded** and halt.

One-phase Simplex Algorithm (1947)

Maximize(LPinstance (A, b, c))

$\{$The instance must be in standard form and origin-feasible$\}$

Construct the feasible dictionary D for the instance

while some x_i has positive coefficient in eq. for z **do**

Find the variable x_j which constrains increasing x_i

$D := D[x_i \leftrightarrow x_j]$

od

return solution corresponding to D

end

Analysis 1: Partial Correctness

We terminate when all variables in the expression for z has negative (or zero) coefficient.

In our basic feasible solution, all these variables are 0. Hence our current solution has value = constant term of z.

It is optimal, as solutions are constrained by $x_1, x_2, \ldots, x_{m+n} \geq 0$.

Analysis 2: Termination

The current dictionary is determined by the variables on its left hand side (as the system of equations is equivalent to the original system of equations).

Thus there are only a finite number of dictionaries.

Furthermore, as the value of z keeps increasing, we will not go back to an earlier dictionary. Thus we terminate.

ANALYSIS INCORRECT!

We do *not* necessarily increase z.

Degenerate Dictionary:

$$\begin{array}{rclllllll} x_3 & = & 0.5 & & & & & - & 0.5x_4 \\ x_5 & = & & - & 2x_1 & + & 4x_2 & + & 3x_4 \\ x_6 & = & & & x_1 & - & 3x_2 & + & 2x_4 \\ z & = & 4 & + & 2x_1 & - & x_2 & - & 4x_4 \end{array}$$

A dictionary is called *degenerate* if some of the coefficients in the "constant-column" are zero.

Staying put

$$\begin{array}{rcrcrcrcr} x_3 & = & 0.5 & & & & & - & 0.5x_4 \\ x_5 & = & & - & 2x_1 & + & 4x_2 & + & 3x_4 \\ x_6 & = & & & x_1 & - & 3x_2 & + & 2x_4 \\ z & = & 4 & + & 2x_1 & - & x_2 & - & 4x_4 \end{array}$$

We must pivot on x_1 and x_5.

$$\begin{array}{rcrcrcrcr} x_1 & = & & & 2x_2 & + & 1.5x_4 & - & 0.5x_5 \\ x_3 & = & 0.5 & & & - & 0.5x_4 & & \\ x_6 & = & & - & x_2 & + & 3.5x_4 & - & 0.5x_5 \\ z & = & 4 & + & 3x_2 & - & x_4 & - & x_5 \end{array}$$

The solution remains the same! *Thus, the simplex algorithm deviates slightly from the local search pattern.*

Cycling is posible

$$\begin{array}{rcrrrrrrrrr}
x_5 & = & & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\
x_6 & = & & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\
x_7 & = & 1 & - & x_1 & & & & & & \\
z & = & & & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4
\end{array}$$

Pivoting on x_1 and x_5, we obtain

$$\begin{array}{rcrrrrrrrrr}
x_1 & = & & & 11x_2 & + & 5x_3 & - & 18x_4 & - & 2x_5 \\
x_6 & = & & - & 4x_2 & - & 2x_3 & + & 8x_4 & + & x_5 \\
x_7 & = & 1 & - & 11x_2 & - & 5x_3 & + & 18x_4 & + & 2x_5 \\
z & = & & & 53x_2 & + & 41x_3 & - & 204x_4 & - & 20x_5
\end{array}$$

Cycling is posible

$$\begin{array}{rcrrrrrrrrr}
x_1 & = & & & 11x_2 & + & 5x_3 & - & 18x_4 & - & 2x_5 \\
x_6 & = & & - & 4x_2 & - & 2x_3 & + & 8x_4 & + & x_5 \\
x_7 & = & 1 & - & 11x_2 & - & 5x_3 & + & 18x_4 & + & 2x_5 \\
z & = & & & 53x_2 & + & 41x_3 & - & 204x_4 & - & 20x_5
\end{array}$$

Pivoting on x_2 and x_6, we obtain

$$\begin{array}{rcrrrrrrrrr}
x_2 & = & & - & 0.5x_3 & + & 2x_4 & + & 0.25x_5 & - & 0.25x_6 \\
x_1 & = & & - & 0.5x_3 & + & 4x_4 & + & 0.75x_5 & - & 2.75x_6 \\
x_7 & = & 1 & + & 0.5x_3 & - & 4x_4 & - & 0.75x_5 & - & 13.25x_6 \\
z & = & & & 14.5x_3 & - & 98x_4 & - & 6.75x_5 & - & 13.25x_6
\end{array}$$

Cycling is posible

$$
\begin{array}{rcrcrcrcrcr}
x_2 & = & & - & 0.5x_3 & + & 2x_4 & + & 0.25x_5 & - & 0.25x_6 \\
x_1 & = & & - & 0.5x_3 & + & 4x_4 & + & 0.75x_5 & - & 2.75x_6 \\
x_7 & = & 1 & + & 0.5x_3 & - & 4x_4 & - & 0.75x_5 & - & 13.25x_6 \\
z & = & & & 14.5x_3 & - & 98x_4 & - & 6.75x_5 & - & 13.25x_6
\end{array}
$$

Pivoting on x_3 and x_1, we obtain

$$
\begin{array}{rcrcrcrcrcr}
x_3 & = & & & 8x_4 & + & 1.5x_5 & - & 5.5x_6 & - & 2x_1 \\
x_2 & = & & - & 2x_4 & - & 0.5x_5 & + & 2.5x_6 & + & x_1 \\
x_7 & = & 1 & & & & & & & - & x_1 \\
z & = & & & 18x_4 & + & 15x_5 & - & 93x_6 & - & 29x_1
\end{array}
$$

Cycling is posible

$$
\begin{array}{rcrrrrrrrr}
x_3 & = & & & 8x_4 & + & 1.5x_5 & - & 5.5x_6 & - & 2x_1 \\
x_2 & = & & - & 2x_4 & - & 0.5x_5 & + & 2.5x_6 & + & x_1 \\
x_7 & = & 1 & & & & & & & - & x_1 \\
z & = & & & 18x_4 & + & 15x_5 & - & 93x_6 & - & 29x_1
\end{array}
$$

Pivoting on x_4 and x_2 we obtain

$$
\begin{array}{rcrrrrrrrrr}
x_4 & = & & - & 0.25x_5 & + & 1.25x_6 & + & 0.5x_1 & - & 0.5x_2 \\
x_3 & = & & - & 0.5x_5 & + & 4.5x_6 & + & 2x_1 & - & 4x_2 \\
x_7 & = & 1 & & & & & - & x_1 & & \\
z & = & & & 10.5x_5 & - & 70.5x_6 & - & 20x_1 & - & 9x_2
\end{array}
$$

Cycling is posible

$$\begin{array}{rcl} x_4 & = & -0.25x_5 + 1.25x_6 + 0.5x_1 - 0.5x_2 \\ x_3 & = & -0.5x_5 + 4.5x_6 + 2x_1 - 4x_2 \\ x_7 & = & 1 - x_1 \\ z & = & 10.5x_5 - 70.5x_6 - 20x_1 - 9x_2 \end{array}$$

Pivoting on x_5 and x_3 we obtain

$$\begin{array}{rcl} x_5 & = & 9x_6 + 4x_1 - 8x_2 - 2x_3 \\ x_4 & = & -x_6 - 0.5x_1 + 1.5x_2 + 0.5x_3 \\ x_7 & = & 1 - x_1 \\ z & = & 24x_6 + 22x_1 - 93x_2 - 21x_3 \end{array}$$

Cycling is posible

$$\begin{array}{rcrrrrrrrrr} x_5 & = & & & 9x_6 & + & 4x_1 & - & 8x_2 & - & 2x_3 \\ x_4 & = & & - & x_6 & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 \\ x_7 & = & 1 & - & & & x_1 & & & & \\ z & = & & & 24x_6 & + & 22x_1 & - & 93x_2 & - & 21x_3 \end{array}$$

Pivoting on x_6 and x_4 we obtain

$$\begin{array}{rcrrrrrrrrr} x_5 & = & & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\ x_6 & = & & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\ x_7 & = & 1 & - & x_1 & & & & & & \\ z & = & & & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4 \end{array}$$

... *and that's where we started.*

Do we care?

Chvatal, page 33: *Cycling is a rare phenomenon [..] For this reason, the remote possibility of cycling is disregarded in most computer implementations of the simplex method*

Does hopefully not apply today......Too many lawyers around!

Bland's rule (1977)

When selecting pivot variables, always select the candidate with the smallest index (e.g., select x_2 over x_4).

Then cycling cannot occur and the simplex algorithm terminates.

Using Anti-cycling rules

We only need to use Blands rule in degenerate situations. If the value of z increases, we are not in a cycle and can pivot in any way we want.

Some pivoting rules to consider in non-degenerate case

- Largest coefficient rule: Let the variable with the largest coefficient in z enter the basis.
- Largest increase rule: Let the variable which will bring about the largest increase in z enter the basis.
- ...

Proof of Bland's rule

Assume we cycle while applying Bland's rule.

$D_1 \rightarrow D_2 \rightarrow \cdots \rightarrow D_l = D_1$

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Let U be indices of the *indecisive variables*, i.e.,
$U = \{r | x_r \text{ basic in some } D_i \text{ and non-basic in some } D_j\}$

Let $t = \max(U)$.

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Let $t = \max(U)$.

Let D be a dictionary in the cycle so that x_t is basic in D but non-basic in the next dictionary.

Let D^* be a dictionary in the cycle so that x_t is non-basic in D^* but basic in the next dictionary.

Proof of Bland's rule

D :

$\{x_i = b_i - \sum_{j \notin B} a_{ij} x_j\}_{i \in B}$ (L)

$z = v + \sum_j c_j x_j \quad (\text{putting } c_j = 0 \text{ for } j \in B)$

Let s be the variable entering the basis in the next step.

Proof of Bland's rule

D :
$\{x_i = b_i - \sum_{j \notin B} a_{ij} x_j\}_{i \in B}$ (L)
$z = v + \sum_j c_j x_j$ (putting $c_j = 0$ for $j \in B$)

Let s be the variable entering the basis in the next step.

Know:

Fact 1: $t \in U \cap B$ and $s \in U - B$.
Fact 2: $c_s > 0$ (as x_s is about to enter the basis).
Fact 3: $a_{ts} > 0$ (as x_t about to leave the basis).
Fact 4: $b_i = 0$ for all $i \in B \cap U$ (as we have a degenerate cycle).
Fact 5: $\forall r \in U \cap B - \{t\} : a_{rs} \leq 0$. (by Bland's rule)

Proof of Bland's rule

D^* :

$\{x_i = b_i^* - \sum_{j \notin B^*} a_{ij}^* x_j\}_{i \in B^*}$

$z = v^* + \sum_j c_j^* x_j \quad \text{(putting } c_j^* = 0 \text{ for } j \in B^*\text{)}$

Proof of Bland's rule

D^* :

$\{x_i = b_i^* - \sum_{j \notin B^*} a_{ij}^* x_j\}_{i \in B^*}$
$z = v^* + \sum_j c_j^* x_j \quad (\text{putting } c_j^* = 0 \text{ for } j \in B^*)$

Know:

Fact 6: $v = v^*$ (as we have a degenerate cycle)
Fact 7: $c_t^* > 0$ (as x_t is about to enter the basis).
Fact 8: $c_r^* \leq 0$ for all $r \in U - \{t\}$. (by Bland's rule)
Fact 9: For any solution x to (L), $\sum_j c_j x_j = \sum_j c_j^* x_j$

Elaborating on Fact 9

$x_s := y$,
$x_j := 0$ for $j \in B^c - \{s\}$,
$x_i := b_i - a_{is}y$ for $i \in B$
is a solution to (L) for any real value y.

Elaborating on Fact 9

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so we have, by Fact 9:

$$\forall y : c_s y = c_s^* y + \sum_{i \in B} c_i^* (b_i - a_{is} y).$$

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so we have, by Fact 9:

$$\forall y : c_s y = c_s^* y + \sum_{i \in B} c_i^* (b_i - a_{is} y).$$

Isolating y, we have

$$\forall y : (c_s - c_s^* + \sum_{i \in B} c_i^* a_{is}) y = \sum_{i \in B} c_i^* b_i$$

Elaborating on Fact 9

$$c_s - c_s^* + \sum_{i \in B} c_i^* a_{is} = 0$$

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$$c_s - c_s^* + \sum_{i \in B} c_i^* a_{is} = 0$$

or, rearranging:

$$\sum_{i \in B} c_i^* a_{is} = c_s^* - c_s$$

Elaborating on Fact 9

$$c_s - c_s^* + \sum_{i \in B} c_i^* a_{is} = 0$$

or, rearranging:

$$\sum_{i \in B} c_i^* a_{is} = c_s^* - c_s$$

$c_s^* - c_s < 0$ by Fact 8 and Fact 2.

The contradiction

$$\sum_{i \in B} c_i^* a_{is} < 0$$

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Fact 11: For some $r \in B$, $c_r^* a_{rs} < 0$.

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$r \notin B^*$ (as $c_r^* \neq 0$, by Fact 11), so $r \in U$.

The contradiction

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Maybe $r = t$?

The contradiction

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The contradiction

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$b_r = 0$ (Fact 4) but in D, x_t leaves the basis and not x_r, so $a_{rs} \leq 0$ (Fact 5).

The contradiction

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In D^*, x_r is non-basic, but is not chosen to enter the basis. So, $c_{r^*} \leq 0$. But then Fact 11 implies that $a_{rs} > 0$.

The contradiction

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In D^*, x_r is non-basic, but is not chosen to enter the basis. So, $c_{r^*} \leq 0$. But then Fact 11 implies that $a_{rs} > 0$.

Oops. . . I mean, woohoo! Our contradiction!

Local Search checklist

Design:

1. How to find first feasible solution. *Done for case of feasible origin*.
2. Define neigborhood structure. *Done - Pivoting*.
3. Strategy for choosing neighbor. *Done - Bland's rule*.

Analysis:

1. Partial correctness (Termination $\Rightarrow$ Correctness). *Done*.
2. Termination. *Done - Bland's rule*
3. Complexity.